SELF-ADJOINT SOLUTION TO THE PROBLEM OF HEAT AND MASS TRANSFER IN A MULTIPHASE MEDIUM

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UDC 536.248.2

A self-adjoint solution is obtained to the problem of heat and mass transfer in a multiphase medium, with the flow of the gaseous phase taken into account.

An analysis of heat and mass transfer processes with phase transformations does, in many cases, require the solution of the following system of differential equations:

for the gaseous phase ($x > x_n(t)$)

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho v}{\partial x}=0,$$
$$c_p\rho\left(\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial x}\right)=\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right),$$
$$\rho\left(\frac{\partial a}{\partial t}+v\frac{\partial a}{\partial x}\right)=\frac{\partial}{\partial x}\left(D\rho\frac{\partial a}{\partial x}\right), \tag{1}$$
$$R\rho T/\mu=p_0^*,$$

for the condensate phase ($x < x_n(t)$)

$$c_k\rho_k\frac{\partial T}{\partial t}=\lambda_k\frac{\partial^2 T}{\partial x^2}.$$

The continuity conditions to be satisfied at the interphase boundary ($x = x_n(t)$) apply to the mass flux of the medium as well as of one of its components, the thermal flux, and the temperature:

$$-\rho_k\frac{dx_n}{dt}=-\rho\frac{dx_n}{dt}+\rho v,$$
$$-\rho_k\frac{dx_n}{dt}=-a\rho\frac{dx_n}{dt}-a\rho v-D\rho\frac{\partial a}{\partial x}, \tag{2}$$
$$\lambda_k\frac{\partial T}{\partial x}=\lambda\frac{\partial T}{\partial x}-\rho_k\frac{dx_n}{dt}L,$$
$$T_{x_n-0}=T_{x_n+0}$$

and valid is also the Clapeyron–Clausius equation

$$a=\exp\left(-\frac{L\mu}{R}\left(\frac{1}{T}-\frac{1}{T_{\text{boil}}}\right)\right).$$

The initial conditions are stipulated as follows:

$$T=\begin{cases}T_0 & \text{at } x>0,\\ T_н & \text{at } x<0,\end{cases}$$
$$a=a_0 \quad \text{at } x>0,\quad \rho=\rho_0 \quad \text{at } x>0,\quad x_n(0)=0.$$

*The assumption of a constant pressure in the heating zone is valid as long as the gas velocity is much lower than the velocity of acoustic perturbations (see [1], for example).

Institute of Physical Chemistry, Academy of Sciences of the USSR, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 24, No. 5, pp. 916-920, May, 1973. Original article submitted July 25, 1972.

The thermal conductivity and the diffusivity are assumed to be power functions of the temperature:

$$\lambda = \lambda_0 (T/T_0)^s, \quad D\rho = D_0\rho_0 (T/T_0)^s.$$

This system describes the process of heat and mass transfer during a momentary contact between a volatile condensate and a hot gas. The subsequent solution is special in that it yields results convenient not only for a preliminary analysis but also for a numerical calculation of the first stage of the process in a more complex system.

System (1) is conveniently analyzed in Langrangian mass coordinates:

$$\tau = t,$$

$$m(x,\ t) = \begin{cases} \rho_k (x - x_n(t)) & \text{at} \quad x < x_n, \\ \int\limits_{x_n}^{x} \rho(\xi,\ t)\, d\xi & \text{at} \quad x > x_n. \end{cases}$$

With $-\rho_k(dx_n/dt)$ denoted by b(t) and $1/\rho$ denoted by V, the system of Eqs. (1)-(2) can be rewritten as follows:

for the gaseous phase (m < 0)

$$\begin{gathered} \frac{\partial V}{\partial \tau} + b\frac{\partial V}{\partial m} - \frac{\partial v}{\partial m} = 0, \\ \frac{\partial T}{\partial \tau} + b\frac{\partial T}{\partial m} = \frac{\partial}{\partial m}\left(\frac{\lambda}{c_p V}\cdot\frac{\partial T}{\partial m}\right), \\ \frac{\partial a}{\partial \tau} + b\frac{\partial a}{\partial m} = \frac{\partial}{\partial m}\left(\frac{D}{V^2}\cdot\frac{\partial a}{\partial m}\right), \\ RT = p_0\mu V, \end{gathered} \qquad (4)$$

for the condensate phase (m > 0)

$$\frac{\partial T}{\partial \tau} + b\frac{\partial T}{\partial m} = \frac{\lambda_k}{c_k V_k}\cdot\frac{\partial^2 T}{\partial m^2}.$$

The continuity conditions at the interphase boundary (m = 0) are

$$\begin{gathered} v = b(V - V_k), \\ b(1-a) + \frac{D}{V^2}\cdot\frac{\partial a}{\partial m} = 0, \\ \frac{\lambda_k}{V_k}\cdot\frac{\partial T}{\partial m} = \frac{\lambda}{V}\cdot\frac{\partial T}{\partial m} - bL, \\ T_{-0} = T_{+0}, \\ a = \exp\left(-\frac{L\mu}{R}\left(\frac{1}{T} - \frac{1}{T_{\text{boil}}}\right)\right). \end{gathered}$$

We will now represent b(τ) and v(m, τ) in the following form:

$$b(\tau) = b_0(\tau)/\sqrt{\tau}, \quad v(m,\ \tau) = v_0(m,\ \tau)/\sqrt{\tau}.$$

It can be easily ascertained that the equations in T, a, V, v_0, b_0 as well as the initial values of these quantities are retained after the similarity transformation:

$$\tau_{\text{new}} = \tau_0 \cdot \tau_{\text{old}}, \quad m_{\text{new}} = m_0 \cdot m_{\text{old}}$$

if $\tau_0 = m_0^2$. Consequently, the solution to problem (4) depends only on the combination of arguments $z = m/\sqrt{\tau}$, which is characteristic of parabolic equations (see [2]). Considering that

$$\frac{\partial}{\partial \tau} = -\frac{z}{2\tau}\cdot\frac{d}{dz}, \quad \frac{\partial}{\partial m} = \frac{1}{\sqrt{\tau}}\cdot\frac{d}{dz},$$

we can easily reduce system (4) to a system of ordinary differential equations.

If we assume that the temperature-dependence (3) of thermal conductivity and of diffusivity corresponds to s = 1 (and such an assumption is more realistic than assuming these quantities to be constant), then the solution to this system can be found in explicit form. Let T_n, a_n, and V_n be respectively the temperature, the concentration, and the specific volume at the interphase boundary, then the solution will be:

for the gaseous phase ($z > 0$)

$$T(z) = T_0 - (T_0 - T_n)\frac{1 - \operatorname{erf}\left(\dfrac{z - 2b_0}{2\sqrt{\lambda_0/c_pV_0}}\right)}{1 + \operatorname{erf}\left(b_0/\sqrt{\lambda_0/c_pV_0}\right)},$$

$$a(z) = a_0 - (a_0 - a_n)\frac{1 - \operatorname{erf}\left(\dfrac{z-2b_0}{2\sqrt{D_0}/V_0}\right)}{1 + \operatorname{erf}\left(b_0V_0/\sqrt{D_0}\right)},$$

$$v_0(z) = b_0(V_n - V_k) + \frac{(T_0 - T_n)}{2T_0}\sqrt{\frac{\lambda_0 V_0}{c_p\pi}}$$

$$\times\frac{\exp\left(-\dfrac{(z - 2b_0)^2}{4\lambda_0/c_pV_0}\right) - \exp\left(\dfrac{b_0^2}{\lambda_0/c_pV_0}\right)}{1 + \operatorname{erf}\left(b_0/\sqrt{\lambda_0/c_pV_0}\right)},$$

for the condensate phase ($z < 0$)

$$T(z) = T_N - (T_N - T_n)\frac{1 + \operatorname{erf}\left(\dfrac{z - 2b_0}{2\sqrt{\lambda_k/c_kV_k}}\right)}{1 - \operatorname{erf}\left(b_0/\sqrt{\lambda_k/c_kV_k}\right)}.$$

The values of T_n, a_n, and V_n necessary for completing the solution will be obtained from the conditions at the interphase boundary:

$$V_n = RT_n/\mu p_0,$$

$$a_n = \exp\left(-\frac{L\mu}{R}\left(\frac{1}{T_n} - \frac{1}{T_{\text{boil}}}\right)\right),$$

$$b_0(1 - a_n) = \frac{\sqrt{D_0}}{2V_0}\,\frac{(a_n - a_0)\exp\left(-b_0^2V_0^2/D_0\right)}{1 + \operatorname{erf}\left(b_0V_0/\sqrt{D_0}\right)},$$

$$\sqrt{\frac{c_k\lambda_k}{\pi V_k}}\cdot\frac{(T_n - T_N)\exp\left(-\dfrac{(z - 2b_0)^2}{4\lambda_k/c_kV_k}\right)}{2\left(1 - \operatorname{erf}\left(b_0/\sqrt{\lambda_k/c_kV_k}\right)\right)}$$

$$= \sqrt{\frac{c_p\lambda_0}{\pi V_0}}\cdot\frac{(T_0 - T_n)\exp\left(-\dfrac{(z-2b_0)^2}{4\lambda_0/c_pV_0}\right)}{2\left(1 + \operatorname{erf}\left(b_0/\sqrt{\lambda_0/c_pV_0}\right)\right)} + b_0L,$$

which reduce to the solution of a transcendental equation in b_0. The following formula is valid for the interphase boundary:

$$x_n(t) = -2b_0\sqrt{t}/\rho_k.$$

A change back to Eulerian coordinates is made by the inverse transformation:

$$x(m, \tau) = x_n(\tau) + m/\rho_k \quad \text{for} \quad m<0,$$

$$x(m, \tau) = x_n(\tau) + \int_0^m V(m, \tau)\,dm \quad \text{for} \quad m>0.$$

This problem arose when the initial stage of the combustion of volatiles was analyzed. Naturally, such a solution can be obtained for an entire class of equation systems describing the process of heat and mass transfer. This method can be used in analyzing a gaseous mixture of many components, for instance, and such an analysis will also apply to a solid–liquid–gas phase transformation.

NOTATION

x is the length coordinate;
t, τ is the time;

ρ	is the density;
c	is the specific heat;
T	is the temperature;
v	is the gas velocity;
λ	is the thermal conductivity;
D	is the diffusivity;
R	is the gas constant;
u	is the molecular weight of the gaseous mixture;
p	is the pressure;
L	is the heat of evaporation;
m	is the mass;
V	is the specific volume;
b	is the velocity of interphase boundary, in mass coordinates;
$z = m/\sqrt{\tau}$;	
s	is the exponent of power-law temperature-dependence of thermal conductivity and diffusivity.

Subscripts

k	refers to the condensate phase;
boil	refers to the boiling point;
old	refers to the old coordinates;
new	refers to the new coordinates.

LITERATURE CITED

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